

Backlund transformations of curves in the Galilean and pseudo-Galilean spaces

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Abstract

Backlund transformations of admissible curves in the Galilean 3-space and pseudo-Galilean 3-space and also spatial Backlund transformations of space curves in Galilean 4-space preserve the torsions under certain assumptions.

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1. Introduction

In the 1890s Bianchi, Lie, and finally Backlund looked at what are now called Backlund transformations of surfaces. In modern parlance, they begin with two surfaces in Euclidean space in a line congruence: there is a mapping between the surfaces M_1 and M_2 such that the line through any two corresponding points is tangent to both surfaces. Backlund proved that if a line congruence satisfied two additional conditions, that the line segment joining corresponding points has constant length, and that the normals at corresponding points form a constant angle, then the two surfaces are necessarily surfaces of constant negative curvature. He was also able to show that a Backlund transformation is integrable, in the sense that given a point on a surface of constant negative curvature and a tangent line segment at that point, a new surface of constant negative curvature can be found, containing the endpoint of the line segment, that is a Backlund transform of the original surface.

The classical Backlund theorem studies the transformation of surfaces of constant negative curvature in \mathbb{R}^3 by realizing them as the focal surfaces of a pseudo-spherical line congruence. The integrability theorem says that we can construct a new surface in \mathbb{R}^3 with constant negative curvature from a given one. In [1] Tenenblat and Terng established a high dimension generalization of Backlund's theorem which is very interesting both for physical and mathematical reasons. After that Chern and Terng customized Backlund theorem for affine

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surfaces [2]. By the same year this transformation was reduced to corresponding asymptotical lines by Terng [3] and following years Tenenblat expanded the Backlund transformation of two surfaces in \mathbb{R}_1^3 to space forms [4]. In 1990 Palmer constructed a Backlund transformation between spacelike and timelike surfaces of constant negative curvature in \mathbb{E}_1^3 [5]. At that decade some researchers gave Backlund transformations on Weingarten surfaces [6–9].

In 1998 Calini and Ivey [10] proposed a geometric realization of the Backlund Transformation for the sine-Gordon equation in the context of curves of constant torsion. Since the asymptotic lines on a pseudospherical surface have constant torsion, the Backlund transformation can be restricted to get a transformation that carries constant torsion curves to constant torsion curves. Later the converse of the idea was proved and generalized for the n -dimensional case by Nemeth [11]. In [12] Nemeth studied a similar concept for constant torsion curves in the 3-dimensional constant curvature spaces. Shief and Rogers used an analogue of the classical Backlund transformation for the generation of soliton surfaces [13]. In [14] Chou, Kouhua and Yongbo obtained the Backlund transformation on timelike surfaces with constant mean curvature in \mathbb{R}_1^2 . Zuo, Chen, Cheng studied Backlund theorems in three dimensional de Sitter space and anti-de Sitter space [15]. Abdel-Baky presented the Minkowski versions of the Backlund theorem and its application by using the method of moving frames [16]. Gürbüz studied Backlund transformations in \mathbb{R}_1^n [17]. Using the same method Özdemir and Çöken have studied Backlund transformations of non-lightlike constant torsion curves in Minkowski 3-space [18].

In this paper we show that a restriction of Backlund theorem on space curves satisfying the given three conditions preserves the torsions of the curves in Galilean and pseudo-Galilean spaces. For the necessary definitions and theorems of Galilean and pseudo-Galilean spaces we referred [20–23]

2. Preliminaries

The Galilean space \mathbb{G}^3 is the three dimensional real affine space with the absolute figure $\{w, f, I\}$, where w is the ideal plane, f is a line in w and I is the fixed elliptic involution of points of f .

The scalar product of two vectors $X = (a_1, a_2, a_3)$ and $Y = (b_1, b_2, b_3)$ in \mathbb{G}^3 is defined by

$$\langle X, Y \rangle_G = \begin{cases} a_1 \cdot b_1 & a_1 \neq 0 \text{ or } b_1 \neq 0 \\ a_2 \cdot b_2 + a_3 \cdot b_3, & a_1 = 0 \text{ and } b_1 = 0 \end{cases}$$

An admissible curve $\alpha : I \subset \mathbb{R} \rightarrow G^3$ of the class C^r ($r \geq 3$) in the Galilean space G^3 is defined by the parametrization

$$\alpha(s) = (s, x(s), y(s))$$

where s is the arc length of α with the differential form $ds = dx$. The curvature $\kappa(s)$ and the torsion $\tau(s)$ of an admissible curve in G^3 are given by $\kappa(s) =$

$\sqrt{y''^2(s) - z''^2(s)}$ and $\tau(s) = (\det(\alpha'(s), \alpha''(s), \alpha'''(s)))/\kappa^2(s)$ respectively. The associated moving trihedron is given by

$$\begin{aligned} E_1 &= \alpha'(s) = (1, x'(s), y'(s)) \\ E_2 &= \frac{(0, x''(s), y''(s))}{\sqrt{x''^2(s) + y''^2(s)}} \\ E_3 &= \frac{(0, -y''(s), x''(s))}{\sqrt{x''^2(s) + y''^2(s)}} \end{aligned}$$

Then the Frenet formulas in the Galilean space G^3 becomes:

$$\begin{aligned} \mathbf{E}'_1 &= \kappa \mathbf{E}_2 \\ \mathbf{E}'_2 &= \tau \mathbf{E}_3 \\ \mathbf{E}'_3 &= -\tau \mathbf{E}_2 \end{aligned} \tag{1}$$

3. Backlund transformations of admissible curves in the Galilean space G^3

Theorem 1. Suppose that ψ is a transformation between two admissible curves α and $\tilde{\alpha}$ in the Galilean space G^3 with $\tilde{\alpha} = \psi(\alpha)$ such that in the corresponding points we have:

- i. The line segment $[\tilde{\alpha}(s)\alpha(s)]$ at the intersection of the osculating planes of the curves has constant length r
- ii. The distance vector $\tilde{\alpha}(s) - \alpha(s)$ has the same angle $\gamma \neq \frac{\pi}{2}$ with the tangent vectors of the curves
- iii. The binormals of the curves have the same constant angle $\phi \neq 0$.

Then these curves are congruent with the curvatures and torsions

$$\begin{aligned} \tilde{\kappa} &= \kappa = -2 \frac{d\gamma}{ds} \\ \tilde{\tau} &= \tau = \frac{\sin \phi}{r} \end{aligned}$$

and the transformation of the curves is given by

$$\tilde{\alpha} = \alpha + \frac{2C}{\tau^2 + C^2} (\cos \gamma \mathbf{E}_1 + \sin \gamma \mathbf{E}_2)$$

where $C = \tau \tan\left(\frac{\phi}{2}\right)$ is a constant and γ is a solution of the differential equation

$$\frac{d\gamma}{ds} = \tau \sin \gamma \tan \frac{\phi}{2}$$

PROOF. Denote by $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ and $(\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{E}}_2, \widetilde{\mathbf{E}}_3)$ the Frenet frames of the curves α and $\tilde{\alpha}$ in the Galilean space \mathbb{G}^3 . Let $\widetilde{\mathbf{E}}_3$ be a unit binormal of $\tilde{\alpha}$.

If we denote by W_1 the unit vector of $\tilde{\alpha} - \alpha$, then we can complete W_1, \mathbf{E}_3 and $W_1, \widetilde{\mathbf{E}}_3$ to the positively oriented orthonormal frames (W_1, W_2, W_3) and $(W_1, \widetilde{W}_2, \widetilde{W}_3)$ where $W_3 = \mathbf{E}_3, \widetilde{W}_3 = \widetilde{\mathbf{E}}_3$ and γ is the angle between W_1 and \mathbf{E}_1 . The frames (W_1, W_2, W_3) and $(W_1, \widetilde{W}_2, \widetilde{W}_3)$ can be obtained by rotating the frames $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ and $(\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{E}}_2, \widetilde{\mathbf{E}}_3)$ around \mathbf{E}_3 and $\widetilde{\mathbf{E}}_3$ with an angle γ respectively. So we can write

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

and

$$\begin{bmatrix} W_1 \\ \widetilde{W}_2 \\ \widetilde{W}_3 \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{E}}_3 \end{bmatrix}.$$

Similarly for a rotation around W_1 by the angle ϕ

$$\begin{aligned} \widetilde{W}_2 &= \cos \phi W_2 - \sin \phi W_3 \\ \widetilde{W}_3 &= \sin \phi W_2 + \cos \phi W_3 \end{aligned}$$

From the above equations we write

$$\begin{aligned} \widetilde{\mathbf{E}}_1 &= (\cos^2 \gamma + \sin^2 \gamma \cos \phi) \mathbf{E}_1 + \cos \gamma \sin \gamma (1 - \cos \phi) \mathbf{E}_2 \\ &\quad + \sin \gamma \sin \phi \mathbf{E}_3 \\ \widetilde{\mathbf{E}}_2 &= \cos \gamma \sin \gamma (1 - \cos \phi) \mathbf{E}_1 + (\sin^2 \gamma + \cos^2 \gamma \cos \phi) \mathbf{E}_2 \\ &\quad - \cos \gamma \sin \phi \mathbf{E}_3 \\ \widetilde{\mathbf{E}}_3 &= -\sin \gamma \sin \phi \mathbf{E}_1 + \sin \phi \cos \gamma \mathbf{E}_2 + \cos \phi \mathbf{E}_3 \end{aligned} \quad (2)$$

Using (1) and (2) for $\widetilde{\mathbf{E}}_3$

$$\begin{aligned} \frac{d\widetilde{\mathbf{E}}_3}{ds} &= -\widetilde{\tau} \widetilde{\mathbf{E}}_2 \\ &= (-\widetilde{\tau} \cos \gamma \sin \gamma (1 - \cos \phi)) \mathbf{E}_1 \\ &\quad + (-\widetilde{\tau} (\sin^2 \gamma + \cos^2 \gamma \cos \phi)) \mathbf{E}_2 \\ &\quad + (\widetilde{\tau} \sin \phi \cos \gamma) \mathbf{E}_3 \end{aligned}$$

and taking derivative of $\widetilde{\mathbf{E}}_3$ in (2) with respect to s

$$\begin{aligned} \frac{d\widetilde{\mathbf{E}}_3}{ds} &= (-\sin \phi \cos \gamma \frac{d\gamma}{ds}) \mathbf{E}_1 \\ &\quad + (-\tau \cos \phi - \sin \gamma \sin \phi (\kappa + \frac{d\gamma}{ds})) \mathbf{E}_2 \\ &\quad + (\tau \sin \phi \cos \gamma) \mathbf{E}_3 \end{aligned}$$

then equating the two statements above we obtain

$$\tilde{\tau} = \tau$$

$$\frac{d\gamma}{ds} = \tau \sin \gamma \tanh \frac{\phi}{2}$$

Similarly, differentiating $\widetilde{\mathbf{E}}_1$ and $\widetilde{\mathbf{E}}_2$ from (2) and using (1)

$$\tilde{\kappa} = \kappa = -2 \frac{d\gamma}{ds}$$

Now α is a unit speed curve. Differentiating

$$r^2 = (\tilde{\alpha} - \alpha)^2$$

and substituting the distance vector

$$\tilde{\alpha} - \alpha = r(\cos \gamma \mathbf{E}_1 + \sin \gamma \mathbf{E}_2) \quad (3)$$

we find that $\tilde{\alpha}$ is also a unit speed curve.

Next taking the derivative of (3) we obtain:

$$\widetilde{\mathbf{E}}_1 = (1 - r \sin \gamma \frac{d\gamma}{ds}) \mathbf{E}_1 + r \cos \gamma (\kappa + \frac{d\gamma}{ds}) \mathbf{E}_2 + \tau r \sin \gamma \mathbf{E}_3$$

From this equation and the Frenet frames (2)

$$\tilde{\tau} = \tau = \frac{\sin \phi}{r}$$

Then rearranging this equality we get

$$r = \frac{2\tau \tan\left(\frac{\phi}{2}\right)}{\tau^2 \left(1 + \tan^2\left(\frac{\phi}{2}\right)\right)}$$

Finally with the aid of (3), naming the constant $C = \tau \tan\left(\frac{\phi}{2}\right)$, the Backlund transformation of the curves is

$$\tilde{\alpha} = \alpha + \frac{2C}{\tau^2 + C^2} (\cos \gamma \mathbf{E}_1 + \sin \gamma \mathbf{E}_2).$$

4. Backlund Transformations of admissible curves in the pseudo-Galilean space \mathbb{G}_1^3

The pseudo-Galilean space \mathbb{G}_1^3 is the three dimensional real affine space with the absolute figure $\{w, f, I\}$, where w is the ideal plane, f is a line in w and I is the fixed hyperbolic involution of the points of f .

The scalar product of two vectors $X = (a_1, a_2, a_3)$ and $Y = (b_1, b_2, b_3)$ in \mathbb{G}_1^3 is defined by

$$\langle X, Y \rangle_G = \begin{cases} a_1 \cdot b_1 & a_1 \neq 0 \text{ or } b_1 \neq 0 \\ a_2 \cdot b_2 - a_3 \cdot b_3, & a_1 = 0 \text{ and } b_1 = 0 \end{cases}$$

The curvature $\kappa(s)$ and the torsion $\tau(s)$ of an admissible curve $\alpha(s) = (s, x(s), y(s))$ in \mathbb{G}_1^3 are given by

$$\kappa(s) = \sqrt{|x''^2(s) - y''^2(s)|} \text{ and } \tau(s) = (\det(\alpha'(s), \alpha''(s), \alpha'''(s)))/\kappa^2(s)$$

respectively. The associated moving trihedron is given by

$$\begin{aligned} E_1 &= \alpha'(s) = (1, x'(s), y'(s)) \\ E_2 &= \frac{(0, x''(s), y''(s))}{\sqrt{|x''^2(s) - y''^2(s)|}} \\ E_3 &= \frac{(0, \varepsilon y''(s), \varepsilon z''(s))}{\sqrt{|x''^2(s) - y''^2(s)|}} \end{aligned}$$

where $\varepsilon = \mp 1$. The Frenet formulas in the pseudo-Galilean space \mathbb{G}_1^3 have the following form:

$$\begin{aligned} \mathbf{E}_1' &= \kappa \mathbf{E}_2 \\ \mathbf{E}_2' &= \tau \mathbf{E}_3 \\ \mathbf{E}_3' &= \tau \mathbf{E}_2 \end{aligned} \tag{4}$$

4.1. Backlund transformations of admissible curves which have time-like binormals in the pseudo-Galilean space \mathbb{G}_1^3 :

Theorem 2. Suppose that ψ is a transformation between two admissible curves α and $\tilde{\alpha}$ in the pseudo-Galilean space \mathbb{G}_1^3 with $\tilde{\alpha} = \psi(\alpha)$ such that in the corresponding points we have:

- i. The line segment $[\tilde{\alpha}(s)\alpha(s)]$ at the intersection of the osculating planes of the curves has constant length r
 - ii. The distance vector $\tilde{\alpha} - \alpha$ has the same angle $\gamma \neq \frac{\pi}{2}$ with the tangent vectors of the curves
 - iii. The timelike binormals of the curves have the same constant angle $\phi \neq 0$.
- Then these curves have equal torsions

$$\tilde{\tau} = \tau = -\frac{\sinh \phi}{r}$$

and the Backlund transformation of the curves is

$$\tilde{\alpha} = \alpha + \frac{2C}{C^2 - \tau^2} (\cos \gamma \mathbf{E}_1 + \sin \gamma \mathbf{E}_2)$$

where $C = \tau \tanh\left(\frac{\phi}{2}\right)$ is a constant and γ is a solution of the differential equation

$$\frac{d\gamma}{ds} = \tau \sin \gamma \tanh \frac{\phi}{2}.$$

PROOF. Denote by $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ and $(\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{E}}_2, \widetilde{\mathbf{E}}_3)$ the Frenet frames of the curves α and $\tilde{\alpha}$ in the pseudo-Galilean space \mathbb{G}_1^3 respectively. Let $\widetilde{\mathbf{E}}_3$ be a unit timelike binormal of $\tilde{\alpha}$ such that $\langle \widetilde{\mathbf{E}}_3, \widetilde{\mathbf{E}}_3 \rangle = -1$. For the rotations of frames we can write

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix},$$

$$\begin{bmatrix} W_1 \\ \widetilde{W}_2 \\ \widetilde{W}_3 \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{E}}_3 \end{bmatrix}$$

and

$$\begin{aligned} \widetilde{W}_2 &= \cosh \phi W_2 + \sinh \phi W_3 \\ \widetilde{W}_3 &= \sinh \phi W_2 + \cosh \phi W_3 \end{aligned}$$

From the equations above we can write

$$\begin{aligned} \widetilde{\mathbf{E}}_1 &= (\cos^2 \gamma + \sin^2 \gamma \cosh \phi) \mathbf{E}_1 + \cos \gamma \sin \gamma (1 - \cosh \phi) \mathbf{E}_2 \\ &\quad - \sin \gamma \sinh \phi \mathbf{E}_3 \\ \widetilde{\mathbf{E}}_2 &= \cos \gamma \sin \gamma (1 - \cosh \phi) \mathbf{E}_1 + (\sin^2 \gamma + \cos^2 \gamma \cosh \phi) \mathbf{E}_2 \\ &\quad + \cos \gamma \sinh \phi \mathbf{E}_3 \\ \widetilde{\mathbf{E}}_3 &= -\sin \gamma \sinh \phi \mathbf{E}_1 + \sinh \phi \cos \gamma \mathbf{E}_2 + \cosh \phi \mathbf{E}_3 \end{aligned} \quad (5)$$

Differentiating $\widetilde{\mathbf{E}}_3$ with respect to the arc length s and using the Frenet equations (4) for $\widetilde{\mathbf{E}}_3$ we find

$$\begin{aligned} \widetilde{\tau} &= \tau \\ \frac{d\gamma}{ds} &= \tau \sin \gamma \tanh \frac{\phi}{2} \end{aligned}$$

Next taking the derivative of the distance vector

$$\tilde{\alpha} - \alpha = r(\cos \gamma \mathbf{E}_1 + \sin \gamma \mathbf{E}_2)$$

and by (5) we get

$$\widetilde{\tau} = \tau = -\frac{\sinh \phi}{r}$$

Then rearranging the equality above

$$r = \frac{2\tau \tanh\left(\frac{\phi}{2}\right)}{\tau^2 \left(\tanh^2\left(\frac{\phi}{2}\right) - 1\right)}$$

Finally with the aid of distance vector, naming the constant $C = \tau \tanh\left(\frac{\phi}{2}\right)$, the Backlund transformation is obtained as

$$\tilde{\alpha} = \alpha + \frac{2C}{C^2 - \tau^2} (\cos \gamma \mathbf{E}_1 + \sin \gamma \mathbf{E}_2)$$

4.2. Backlund transformations of admissible curves which have time-like normals in the pseudo-Galilean space \mathbb{G}_1^3 :

Theorem 3. Suppose that ψ is a transformation between two admissible curves α and $\tilde{\alpha}$ in the pseudo-Galilean space \mathbb{G}_1^3 with $\tilde{\alpha} = \psi(\alpha)$ such that in the corresponding points we have:

i. The line segment $[\tilde{\alpha}(s)\alpha(s)]$ at the intersection of the osculating planes of the curves has constant length r

ii. The distance vector $\tilde{\alpha} - \alpha$ has the same angle $\gamma \neq 0$ with the tangent vectors of the curves

iii. The timelike normals of the curves have the same constant angle $\phi \neq 0$. Then these curves have the relation between their torsions

$$\tilde{\tau} = -\tau = -\frac{\sinh \phi}{r}$$

and the Backlund transformation of the curves is given by

$$\tilde{\alpha} = \alpha + \frac{2C}{C^2 - \tau^2} (\cosh \gamma \mathbf{E}_1 + \sinh \gamma \mathbf{E}_2)$$

where $C = \tau \tanh \left(\frac{\phi}{2} \right)$ is a constant and γ is a solution of the differential equation

$$\frac{d\gamma}{ds} = -\tau \sinh \gamma \tanh \frac{\phi}{2}$$

PROOF. Denote by $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ and $(\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{E}}_2, \widetilde{\mathbf{E}}_3)$ the Frenet frames of the curves α and $\tilde{\alpha}$ in the pseudo-Galilean space \mathbb{G}_1^3 respectively. Let $\widetilde{\mathbf{E}}_2$ be a unit timelike normal of $\tilde{\alpha}$ such that $\langle \widetilde{\mathbf{E}}_2, \widetilde{\mathbf{E}}_2 \rangle = -1$.

Again by the notation of previous proof it can be written

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} \cosh \gamma & \sinh \gamma & 0 \\ \sinh \gamma & \cosh \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix},$$

$$\begin{bmatrix} \widetilde{W}_1 \\ \widetilde{W}_2 \\ \widetilde{W}_3 \end{bmatrix} = \begin{bmatrix} \cosh \gamma & \sinh \gamma & 0 \\ \sinh \gamma & \cosh \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{E}}_3 \end{bmatrix}$$

and

$$\begin{aligned} \widetilde{W}_2 &= \cosh \phi W_2 + \sinh \phi W_3 \\ \widetilde{W}_3 &= -\sinh \phi W_2 + \cosh \phi W_3 \end{aligned}$$

From the above equations we write

$$\begin{aligned} \widetilde{\mathbf{E}}_1 &= (\cosh^2 \gamma - \sinh^2 \gamma \cosh \phi) \mathbf{E}_1 + \cosh \gamma \sinh \gamma (1 - \cosh \phi) \mathbf{E}_2 \\ &\quad - \sinh \gamma \sinh \phi \mathbf{E}_3 \\ \widetilde{\mathbf{E}}_2 &= \cosh \gamma \sinh \gamma (-1 + \cosh \phi) \mathbf{E}_1 + (-\sinh^2 \gamma + \cosh^2 \gamma \cosh \phi) \mathbf{E}_2 \quad (6) \\ &\quad + \cosh \gamma \sinh \phi \mathbf{E}_3 \\ \widetilde{\mathbf{E}}_3 &= -\sinh \gamma \sinh \phi \mathbf{E}_1 - \sinh \phi \cosh \gamma \mathbf{E}_2 + \cosh \phi \mathbf{E}_3 \end{aligned}$$

Differentiating $\widetilde{\mathbf{E}}_3$ with respect to the arc length s and using Frenet equation for $\widetilde{\mathbf{E}}_3$ we find

$$\begin{aligned}\widetilde{\tau} &= -\tau \\ \frac{d\gamma}{ds} &= -\tau \sinh \gamma \tanh \frac{\phi}{2}\end{aligned}$$

Next taking the derivative of the distance vector

$$\widetilde{\alpha} - \alpha = r(\cosh \gamma \mathbf{E}_1 + \sinh \gamma \mathbf{E}_2)$$

and from (6) it can be found

$$\widetilde{\tau} = -\tau = \frac{\sinh \phi}{r}$$

Then rearranging the equality above we get

$$r = \frac{2\tau \tanh\left(\frac{\phi}{2}\right)}{\tau^2 \left(\tanh^2\left(\frac{\phi}{2}\right) - 1\right)}$$

Finally with the aid of distance vector, naming the constant $C = \tau \tanh\left(\frac{\phi}{2}\right)$, the transformation is obtained as

$$\widetilde{\alpha} = \alpha + \frac{2C}{C^2 - \tau^2}(\cosh \gamma \mathbf{E}_1 + \sinh \gamma \mathbf{E}_2)$$

5. Spatial Backlund transformations of curves in Galilean space \mathbb{G}^4

The Galilean space \mathbb{G}^4 consists of a four dimensional real affine space endowed with global absolute time and Euclidean metric structure E over the simultaneity hyperplanes defined as the three-dimensional real affine spaces with underlying vector space $\text{Ker}(t)$ of the absolute time functional which is a non zero linear functional $t : V \rightarrow \mathbb{R}$ on the underlying vector space V of E .

The scalar product of two vectors $X = (a_1, a_2, a_3, a_4)$ and $Y = (b_1, b_2, b_3, b_4)$ in \mathbb{G}^4 is defined by

$$\langle X, Y \rangle_G = \begin{cases} a_4 \cdot b_4 & a_4 \neq 0 \text{ or } b_4 \neq 0 \\ a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3, & a_4 = 0 \text{ and } b_4 = 0 \end{cases}$$

Let $\alpha(s) = (x(s), y(s), z(s), t(s))$ be the position vector of a curve. Then the condition $\alpha' = E_1, |E_1| = 1$ is equivalent to the condition $t(s) = s$. Thus natural equations of a curve $\alpha(s) = (x(s), y(s), z(s), s)$ in G^4 are

$$\begin{aligned}\kappa(s) &= \sqrt{x''^2(s) + y''^2(s) + z''^2(s)} \\ \tau(s) &= (\det(\alpha'(s), \alpha''(s), \alpha'''(s)) / \kappa^2(s).\end{aligned}$$

The unit tangent vector, the unit normal vector, the unit binormal vector and the temporal vector (of the time axis) of the curve are shown by E_1, E_2, E_3, E_4 respectively. Thus

$$\begin{aligned} E_1 &= \alpha'(s) = (x'(s), y'(s), z'(s), 1) \\ E_2 &= \frac{E_1'}{\kappa(s)} \\ E_3 &= \frac{E_2'}{\tau(s)} \\ E_4 &= \mu E_1 \wedge E_2 \wedge E_3 \end{aligned}$$

where μ is chosen as ± 1 for $\det(E_1, E_2, E_3, E_4)$ to be 1.

The Frenet equations in the Galilean 4-space with the spatial Frenet vectors E_1, E_2, E_3 and the temporal vector E_4 are given by

$$\begin{aligned} \mathbf{E}_1' &= \kappa \mathbf{E}_2 \\ \mathbf{E}_2' &= -\kappa \mathbf{E}_1 + \tau \mathbf{E}_3 \\ \mathbf{E}_3' &= -\tau \mathbf{E}_2 \\ \mathbf{E}_4' &= -\sigma \mathbf{E}_3 \end{aligned} \tag{7}$$

Galilean geometry is the study of properties of figures that are invariant under the Galilean transformations. In general a Galilean transformation in n spatial dimensions takes the $(n+1)$ -vector (\mathbf{u}, t) to the $(n+1)$ -vector $(R\mathbf{u} + vt + a, t + a_0)$ where $R \in SO(n), v \in R^n$ and $a \in R^n$. Particularly in \mathbb{G}^4 , a spatial rotation of reference frame happens for the plane spanned by two spatial axes holding the other plane stationary.

5.1. Spatial Backlund transformations of curves in the Galilean 4-space:

Theorem 4. Suppose that ψ is a transformation between two curves α and $\tilde{\alpha}$ in the Galilean space \mathbb{G}^4 with $\tilde{\alpha} = \psi(\alpha)$. We have:

- i. The line segment $[\tilde{\alpha}(s)\alpha(s)]$ at the intersection of the osculating planes of the curves has constant length r
- ii. The distance vector $\tilde{\alpha} - \alpha$ has the same Euclidean angle $\gamma \neq \frac{\pi}{2}$ with the tangent vectors of the curves
- iii. The binormals of the curves have the same constant Euclidean angle $\phi \neq 0$.

Then curvatures, torsions and the spatial Backlund transformation of the curves are given by

$$\begin{aligned} \tilde{\kappa} &= -\kappa - 2\frac{d\gamma}{ds} \\ \tilde{\tau} &= \tau = \frac{\sin \phi}{r} \\ \tilde{\alpha} &= \alpha + \frac{2C}{\tau^2 + C^2}(\cos \gamma \mathbf{E}_1 + \sin \gamma \mathbf{E}_2) \end{aligned}$$

where the Backlund parameter is $C = \tau \tan\left(\frac{\phi}{2}\right)$ and γ is a solution of the differential equation $\frac{d\gamma}{ds} = \tau \sin \gamma \tan\left(\frac{\phi}{2}\right) - \kappa$.

PROOF. Denote by $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4)$ and $(\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{E}}_2, \widetilde{\mathbf{E}}_3, \widetilde{\mathbf{E}}_4)$ the Frenet frame of the curves α and $\tilde{\alpha}$ in the Galilean space \mathbb{G}^4 respectively. Let $\widetilde{\mathbf{E}}_3$ be the unit binormal of $\tilde{\alpha}$.

If we denote by W_1 the unit vector of $\tilde{\alpha} - \alpha$, then we can complete $W_1, \mathbf{E}_3, \mathbf{E}_4$ and $W_1, \widetilde{\mathbf{E}}_3, \widetilde{\mathbf{E}}_4$ to the positively oriented orthonormal frames (W_1, W_2, W_3, W_4) and $(W_1, \widetilde{W}_2, \widetilde{W}_3, \widetilde{W}_4)$ where $W_3 = \mathbf{E}_3, \widetilde{W}_3 = \widetilde{\mathbf{E}}_3, W_4 = \mathbf{E}_4, \widetilde{W}_4 = \widetilde{\mathbf{E}}_4$. For a spatial rotation of the $E_1 E_2$ plane holding the other plane constant we can write

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 & 0 \\ -\sin \gamma & \cos \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix}$$

and similarly for $\widetilde{E}_1 \widetilde{E}_2$ plane

$$\begin{bmatrix} W_1 \\ \widetilde{W}_2 \\ \widetilde{W}_3 \\ \widetilde{W}_4 \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 & 0 \\ -\sin \gamma & \cos \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{E}}_3 \\ \widetilde{\mathbf{E}}_4 \end{bmatrix}.$$

Also we can rotate spatially the $\widetilde{W}_2 \widetilde{W}_3$ plane by the transformation

$$\begin{bmatrix} W_1 \\ \widetilde{W}_2 \\ \widetilde{W}_3 \\ \widetilde{W}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi & 0 \\ 0 & \sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}$$

From the above equations we find the Frenet vectors

$$\begin{aligned} \widetilde{\mathbf{E}}_1 &= (\cos^2 \gamma + \sin^2 \gamma \cos \phi) \mathbf{E}_1 + \cos \gamma \sin \gamma (1 - \cos \phi) \mathbf{E}_2 \\ &\quad + \sin \gamma \sin \phi \mathbf{E}_3 \\ \widetilde{\mathbf{E}}_2 &= \cos \gamma \sin \gamma (1 - \cos \phi) \mathbf{E}_1 + (\sin^2 \gamma + \cos^2 \gamma \cos \phi) \mathbf{E}_2 \\ &\quad - \cos \gamma \sin \phi \mathbf{E}_3 \\ \widetilde{\mathbf{E}}_3 &= -\sin \gamma \sin \phi \mathbf{E}_1 + \sin \phi \cos \gamma \mathbf{E}_2 + \cos \phi \mathbf{E}_3 \\ \widetilde{\mathbf{E}}_4 &= \mathbf{E}_4 \end{aligned} \tag{8}$$

Since α is a unit speed curve, differentiating the below

$$\|\tilde{\alpha} - \alpha\|^2 = \langle \tilde{\alpha} - \alpha, \tilde{\alpha} - \alpha \rangle = r^2$$

and substituting the distance vector

$$\tilde{\alpha} - \alpha = r(\cos \gamma \mathbf{E}_1 + \sin \gamma \mathbf{E}_2)$$

we find that $\tilde{\alpha}$ is also a unit speed curve. By the derivative of above equation and of $\tilde{\mathbf{E}}_1$ with respect to the arclength s we find

$$\begin{aligned}\tau &= \frac{\sin \phi}{r} \\ \frac{d\gamma}{ds} &= \tau \sin \gamma \tan \left(\frac{\phi}{2} \right) - \kappa\end{aligned}$$

Also from the derivative of $\tilde{\mathbf{E}}_3$ and use of Frenet equations gives

$$\tilde{\tau} = \tau$$

A similar approach for $\tilde{\mathbf{E}}_1$ results in the equality

$$\tilde{\kappa} = -\kappa - 2 \frac{d\gamma}{ds}$$

Hence the transformation of the curves becomes

$$\tilde{\alpha} = \alpha + \frac{2C}{\tau^2 + C^2} (\cos \gamma \mathbf{E}_1 + \sin \gamma \mathbf{E}_2)$$

with $C = \tau \tan(\frac{\phi}{2})$.

6. References

References

- [1] K. Tenenblat, C.L. Terng, Bäcklund's theorem for n-dimensional submanifolds in \mathbb{R}^{2n-1} , Ann. Math. 111 (1980) 477-490.
- [2] S.S.Chern, C.L.Terng, An analogue of Bäcklund's theorem in affine geometry, Rocky Mt. J. Math. 10 (1980) 105-124.
- [3] C.L. Terng, A higher dimension generalization of the Sine-Gordon equation and its soliton theory, Ann. Math. 111 (1980) 491-510.
- [4] K. Tenenblat, Bäcklund theorems for submanifolds of space forms and a generalized wave equation, Bull. Soc. Brasil. Mat. 16 (1985) 67-92.
- [5] B. Palmer, Bäcklund transformations for surfaces in Minkowski space, J. Math. Phys. 31 (1990) 2872-2875.
- [6] S.G. Buyske, Geometric aspects of Backlund transformations of Weingarten submanifolds, Pacific J. Math. 166 (1994) 213-223.
- [7] T. Chou, C. Xifang, Bäcklund transformation on surfaces with $aK+bH=c$, Chin. J. Contemp. Math. 18 (1997) 353-364.
- [8] W. Chen, H. Li, Weingarten surfaces and sine-Gordon equation, Sci. China Ser. A, 40 (1997) 1028-1035.

- [9] C. Xifang, T. Chou, Bäcklund transformations on surfaces with $(k_1 - m)(k_2 - m) = \pm l^2$ in $\mathbf{R}^{2,1}$, J. Phys. A: Math. Gen, 30 (1997) 6009.
- [10] A. Calini, T. Ivey, Bäcklund transformations and knots of constant torsion, J. Knot Theor. Ramif. 7 (1998) 719-746.
- [11] S.Z. Németh, Bäcklund transformations of n-dimensional constant torsion curves, Publ. Math-Debrecen 53 (1998) 271-279.
- [12] S.Z. Németh, Bäcklund transformations of constant torsion curves in 3-dimensional constant curvature spaces, Ital. J. Pure Appl. Math. 7 (2000) 125-138.
- [13] W.K. Schief, C. Rogers, Binormal motion of curves of constant curvature and torsion. Generation of soliton surfaces, P. Roy. Soc. Lond. A Mat. 455 (1999) 3163-3188
- [14] T. Chou, Z. Kouhwa, T. Yongbo, Bäcklund transformation on surfaces with constant mean curvature in $\mathbf{R}^{2,1}$, 23B (2003) 369-376
- [15] D. Zuo, Q. Chen, Y. Cheng, Bäcklund theorems in three-dimensional de Sitter space and anti-de Sitter space, J. Geom. Phys. 44 (2002) 279-298.
- [16] R.A. Abdel Baky, The Backlund's theorem in Minkowski 3-space \mathbf{R}_1^3 , Appl. Math. Comput. 160 (2005) 41-50.
- [17] N. Gürbüz, Backlund transformations of constant torsion curves in \mathbf{R}_1^n , Hadronic J. 29 (2006) 213-220.
- [18] M. Özdemir, A.C. Çöken, Bäcklund transformation for non-lightlike curves in Minkowski 3-space, Chaos Solitons Fract. 42 (2009) 2540-2545.
- [19] M. Akyiğit, A.Z. Azak, M. Tosun, Admissible Mannheim curves in pseudo-Galilean space \mathbf{G}_1^3 , <http://arxiv.org/abs/1001.2440v3>, 2010
- [20] I.M. Yaglom, A simple non-Euclidean Geometry and its physical basis, Springer-Verlag, New York, 1979
- [21] S. Yilmaz, Construction of the Frenet-Serret frame of a curve in 4D Galilean space and some applications, Int. J. Phys. Sci. 5 (2010) 1284-1289.
- [22] B. Divjak, The general solution of the Frenet system of differential equations for curves in the pseudo-Galilean space \mathbf{G}_1^3 , Math. Commun. 2 (1997) 143-147.
- [23] B.J. Pavkovic, I. Kamenarovic, The equiform differential geometry of curves in the Galilean space \mathbf{G}^3 , Glasnik Mat. 22 (1987) 449-457.